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COMPUTATIONAL EXPERIMENTS ON INTERACTIONS BETWEEN NUMERICAL AND PHYSICAL INSTABILITIES

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SUMMARY

Three numerical examples of singular flows in two and three dimensions computed with the vortex method are presented. The effect of the cut-off parameter is investigated and special techniques are added to the classical vortex method to diminish the numerical instabilities present in the examples.

KEY WORDS Vortex methods Kelvin–Helmholtz instabilities

1. INTRODUCTION

Many interesting numerical experiments have been carried out over the past decade in which two- and three-dimensional incompressible flows are calculated by vortex point methods. In these methods the initial vorticity distribution is discretized by a finite number of points or blobs corresponding to a uniform mesh. The motion of these points is described by computing their interaction[1,2] according to the Biot–Savart law. In a recent work on three-dimensional flows[3] it has been pointed out that perturbations induced by computer round-off errors lead to an irregular motion of these points and to the development of widely unstable modes. These phenomena have also been observed by Krasny[4] when studying Kelvin–Helmholtz instabilities. The conclusion of Krasny's work can be summarized as follows. The Kelvin–Helmholtz problem is ill-posed but a smooth solution can be computed by particular techniques. The goal of such techniques is to avoid perturbations due to round-off error. Herein lies the main difference from the present work since we do not try to remove such perturbations. We try instead to evaluate if they are equivalent to physical perturbations arising in real flows. The problem to be considered here is to find out whether the evolution of these perturbations follows a physical rather than a numerical path. This paper reports on some computational experiments which were performed in order to study this problem. We consider three types of flow: evolution of a random vorticity distribution, the Kelvin–Helmholtz problem and three-dimensional vortex ring flow.

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2. VORTEX METHOD

Let $u(x, t)$ be the velocity field and $\omega = \mathrm{curl}\,(u)$ the vorticity field. The vorticity transport equation is given by

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega + (\omega \cdot \nabla)u = 0, \tag{1}$$

$$u = K^* \omega, \tag{2}$$

where K is the curl of the Green kernel in the whole space ($\mathbb{R}^2$ or $\mathbb{R}^3$). A particle of fluid located at $x = x(t)$ moves with the fluid and the vorticity $\omega(x(t), t)$ is stretched according to the following equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u(x(t), t), \tag{3}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = [\nabla u] \cdot \omega. \tag{4}$$

Then the vortex method consists simply of approximating equations (3) and (4) by representing the vorticity distribution as a linear combination of Dirac measures:

$$\omega^{\mathrm{h}} = \sum_i \alpha_i \delta(x - x_i).$$

Looking at the discretized form of equation (2), it is obvious that this kind of description gives rise to singular flows and thus the kernel K must be smoothed.[1] The simplest smoothing results from a regularization of the velocity: K is replaced by $K_\varepsilon = K^* \zeta_\varepsilon$, where $\zeta_\varepsilon = \varepsilon^{-n} \zeta(x/\varepsilon)$ ($n = 2$ or 3) and ζ is a convenient smoothing function. If we denote by $x_i(t)$ the location of particle i and by $\alpha_i(t)$ the weight of particle i, then the approximate vorticity and velocity are given by

$$\omega^{\mathrm{h}}(x, t) = \sum_i \alpha_i(t) \delta(x - x_i)(t)),$$

$$u^{\mathrm{h}}(x, t) = \sum_i K_\varepsilon(x - x_i)(t))\alpha_i(t).$$

Finally we obtain a system of differential equations for the particle locations and weights by approximating equation (3) and (4):

$$\frac{\mathrm{d}x_k}{\mathrm{d}t} = \sum_i K_\varepsilon(x_k(t) - x_i(t))\alpha_i(t), \tag{5}$$

$$\frac{\mathrm{d}\alpha_k}{\mathrm{d}t} = \alpha_k \nabla\left(\sum_i K_\varepsilon(x_k(t) - x_i(t))\alpha_i(t) \right). \tag{6}$$

For more details see References 1 and 2.

3. 2-D CASE

Non-smooth flow

This section is devoted to an example already published in Reference 5. The example is reconsidered from the specific point of view of the present paper.

In the 2D case the vorticity is a scalar and there is no stretching. Thus equation (4) reduces to

$$\frac{\mathrm{d}\omega}{\mathrm{d}t}=0,$$

the discretized form of which is

$$\frac{\mathrm{d}\alpha_k}{\mathrm{d}t}=0.$$

This equation implies that the weight of the particle is constant. Then the method reduces to the approximation of equation (3) only. The convergence has been proved for $\omega \in L^1 \cap L^\infty$.[3]

The first case we consider is the time evolution of an initial vorticity distribution which is highly oscillating without any typical length scale. For this purpose we choose a uniform distribution of particles in a square, together with a random initialization of the weights α_i. Regularization of the velocity smooths the flow and introduces a characteristic length scale ε (ε represents the domain of interaction of a particle). It might be assumed that a small value of this parameter would give rise to an accurate representation of all the length scale greater than ε (restricted by the number of discretization points). However, this is not the case (except for small time, depending on ε) because the algorithm is not stable (either from a theoretical or a numerical point of view[3]). We must increase the value of this parameter and the numerical solution then represents some mean flow which is not an accurate representation of the exact solution. In order to improve the accuracy of the algorithm, a special technique has been proposed by Beale.[6] The effect of this technique has been explored in References 6 and 7 and it is natural to try to use it here. It consists of modifying the weights of the particles in order to obtain correct values of the vorticity at the particle locations. This is done by solving the following system at each time step:

$$\alpha_i=h^2\sum_k \gamma_k \zeta_\varepsilon(x_i-x_k), \tag{7}$$

where h is the particle path and (x_k) are the particle locations at the time considered.

Since ζ_ε is an approximation of identity, the linear system (7) is well-conditioned. Thus we can use an iterative method to solve it:

$$\gamma_i^n=\alpha_i-\gamma_i^{n-1}+h^2\sum_k \gamma_k^{n-1}\zeta_\varepsilon(x_i-x_k). \tag{8}$$

The coefficients (γ_k) obtained after n iterations are used to compute the velocity:

$$u^{\mathrm{h}}=\sum_i K_\varepsilon(x-x_i)\gamma_i^n. \tag{9}$$

This method works well in the case of smooth flows with only a few iterations,[6,7] but in the present situation it does not. We think that the reason for this is that the vorticity is not conserved:

$$\sum_i \alpha_i \neq \sum_k \gamma_k^n.$$

Also, it is possible to construct a conservative form of this algorithm with a simple modification:

$$\gamma_i^n=\alpha_i+h^2\sum_k (\gamma_k^{n-1}-\gamma_i^{n-1})\zeta_\varepsilon(x_i-x_k). \tag{10}$$

This new formulation is found to be equivalent to the introduction of artificial viscosity,[5] and equation (10) can be interpreted as an anti-diffusive process which enhances the numerical instability of small scales. Therefore it is possible to use a similar form in order to obtain an artificial diffusion algorithm which will damp the instability:

$$\gamma_i^n = \alpha_i + \sigma \sum_k (\gamma_k^{n-1} - \gamma_i^{n-1}) \zeta_\varepsilon (x_i - x_k),$$

with σ negative.

In Figure 1 we present a comparison of the entropy computed with the classical scheme (constant weights) and with the present diffusive and anti-diffusive techniques. We first see that with the classical scheme the entropy grows. As expected, the effect of the artificial diffusion is to smooth the vorticity so that the computed solution represents the mean flow. In this case the algorithm is stable. In contrast, with anti-diffusion we do not represent the mean flow and the entropy increases with time.

We may emphasize a distinction between smoothing, which acts on the velocity fields, and diffusion (or anti-diffusion), which affects the vorticity fields.

In Figure 2 we present the same computation but with a smooth initial vorticity field. The differences between the three different cases are not really significant. It can be concluded that the correction acts on scales that are small compared with the parameter ε.

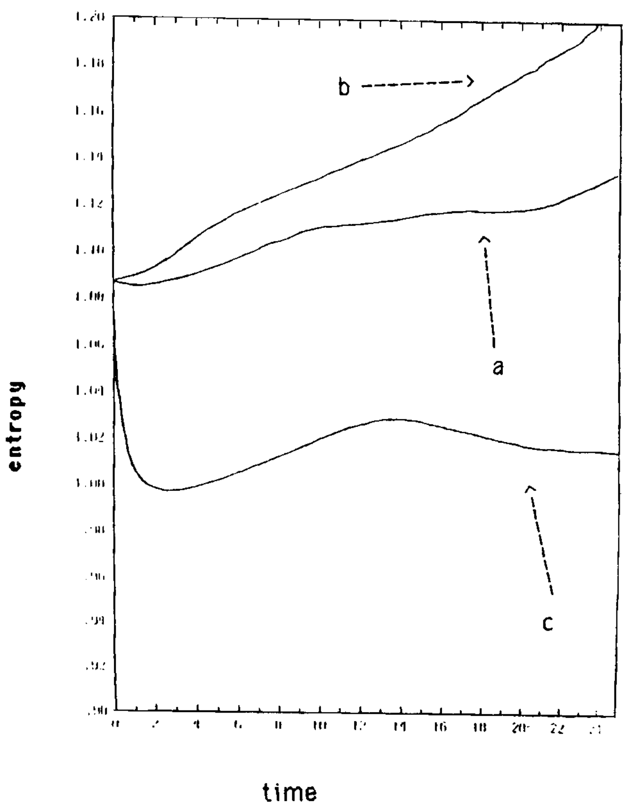


Figure 1. Illustration of the different schemes ($\varepsilon = 0{\cdot}2$): (a) constant weight; (b) anti-diffusive scheme; (c) numerical viscosity model

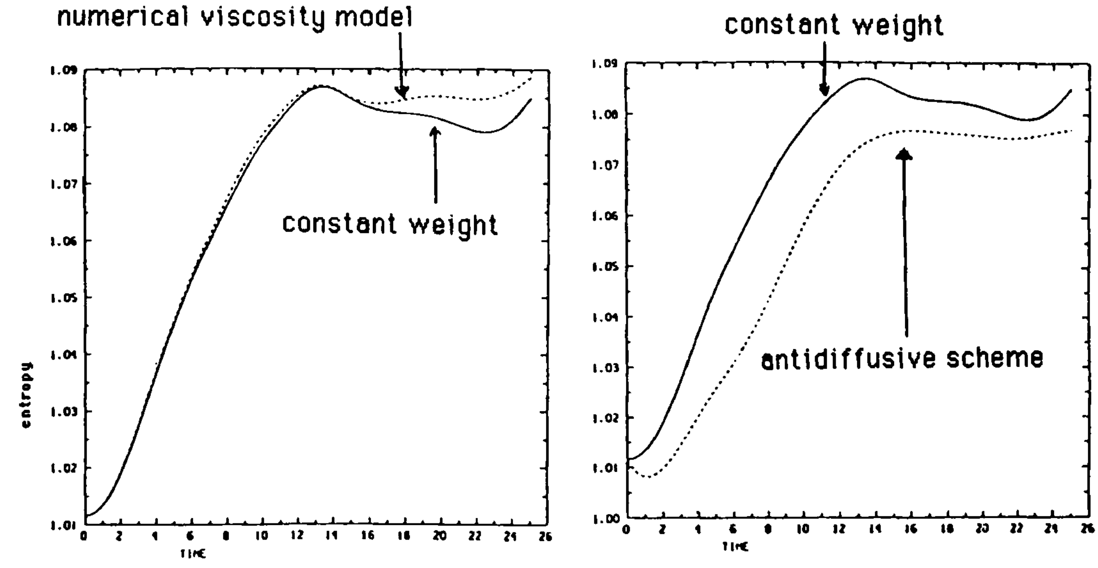


Figure 2. Effect of the different schemes in the case of smooth flow

Kelvin–Helmholtz instabilities

The second test involves Kelvin–Helmholtz instabilities. It has been shown[4] that perturbations induced by computer round off errors lead to irregular motion. A computation will be realistic if this effect is balanced. This source of errors can be controlled by the use of a high-precision machine or a filtering technique.[4]

A vortex sheet in 2D can be represented by a function $z=z(\Gamma, t)$, where t is the time, z is the space variable and Γ is a Lagrangian parameter which measures the circulation. We introduce two perturbations of different wavelengths: $z=\Gamma+\sigma p(\Gamma)+\sigma_1 p(\Gamma/l_1)$, where p is a periodic function of period 1. Then the regularized velocity $U=(u, v)$ can be written

$$u=\frac{1}{2}\int\frac{-\sinh[2\pi(y-y')]}{\cosh[2\pi(y-y')]-\cosh(2\pi(x-x')]+\varepsilon^2}\,\mathrm{d}\Gamma',$$

$$v=\frac{1}{2}\int\frac{\sin[2\pi(x-x')]}{\cosh[2\pi(y-y')]-\cosh[2\pi(x-x')]+\varepsilon^2}\,\mathrm{d}\Gamma'.$$

In Figure 3 we present a classical roll-up of a vortex sheet ($\sigma=0{\cdot}01$, $\sigma_1=0$) similar to those presented in Krasny's work. We observe a spiral with a finite number of turns; the number of turns of the computed solution depends on ε.

In the case where two modes are present it will be interesting to see if the evolution of these modes is perturbed by computer round-off errors. We see in Figure 4(a) that the highest one develops more rapidly, in agreement with stability analysis. Moreover, the sheet cannot be decomposed into two independent curves. This shows that the two modes interact through non-linear effects. This can be also seen in Figure 4(b) where we have plotted the amplitudes of the Fourier coefficients of the sheet for a high-precision machine and with a filtering technique ($\sigma=0{\cdot}01$, $\sigma_1=0{\cdot}0001$, $l_1=0{\cdot}01$). In this case the initial condition has non-zero Fourier coefficients for modes $k=\pm 1$, ± 100, but round-off errors are present in both cases. The effect of the filter is

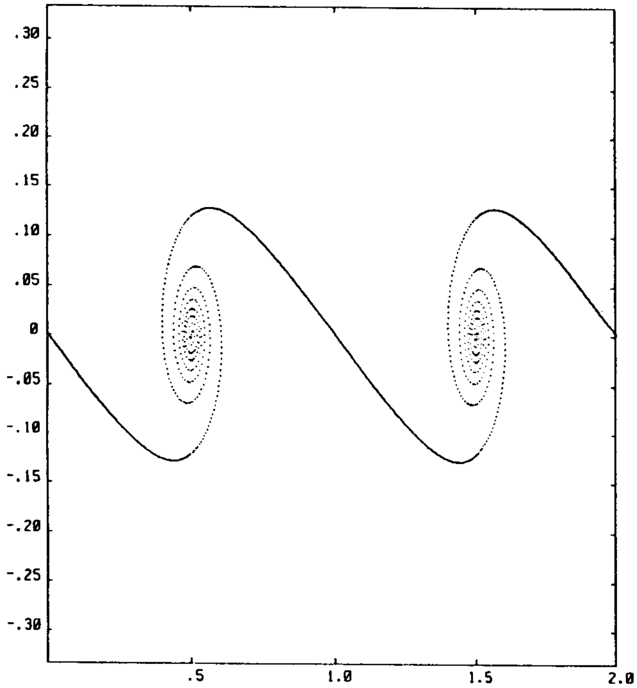


Figure 3. Classical roll-up of a vortex sheet at time 1·1 ($\sigma = 0{\cdot}01$, $\sigma_1 = 0$)

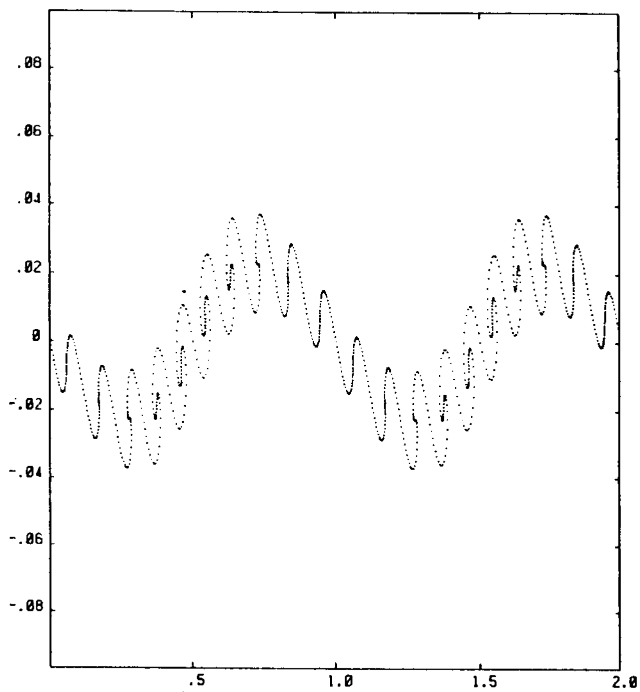
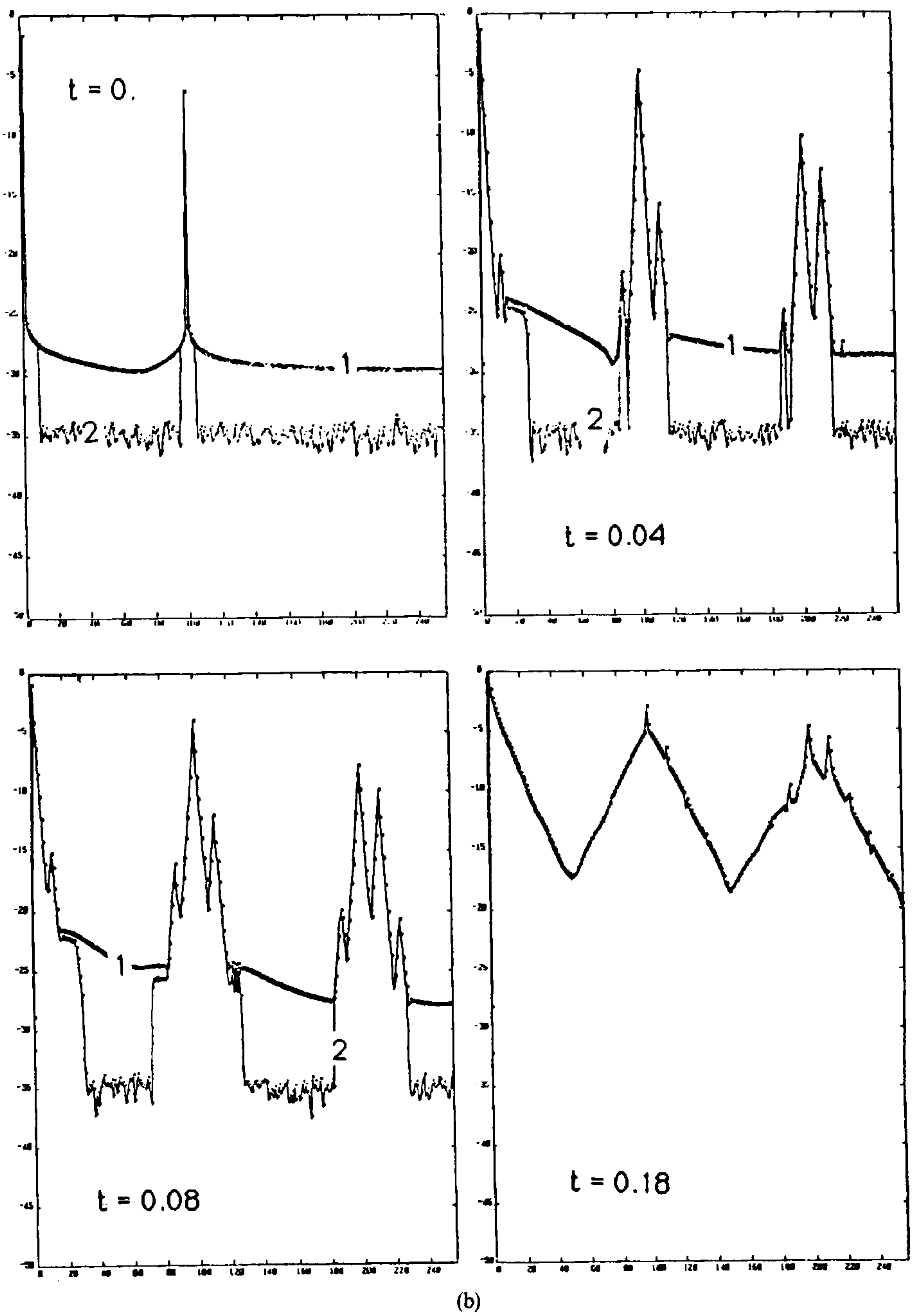


Figure 4(a)



(b)

Figure 4. (a) Point positions at time 0·4 for the case of two modes ($\varepsilon=0{\cdot}05$, $\sigma=0{\cdot}01$, $\sigma_1=0{\cdot}0001$, $l_1=0{\cdot}1$). (b) Log–linear plots of the amplitudes of the Fourier coefficients ($\varepsilon=0{\cdot}05$, $\sigma=0{\cdot}01$, $\sigma_1=0{\cdot}0001$, $l_1=0{\cdot}01$) for a high-precision machine (curve 1) and with a filtering technique (curve 2)

not to suppress the growth of the Fourier modes but to damp the noise due to round-off errors. We observe that, at later time, when the amplitude of a mode has jumped above the filter level, it follows, the evolution of the unfiltered one. Moreover, the same new modes appear in both cases. We can conclude then that perturbations induced by computer round-off do not seem to perturb the evolution of the dominant modes and that filtering does not prevents the appearance of new modes through non-linear effects.

Another way to control the spurious growth of short-wavelength modes is to adjust the parameter ε. If we decrease it, small-wavelength perturbations develop, resulting in a large error in the computed curve (Figure 5(a)). This is also clear in Figure 5(b) where the amplitudes of the Fourier coefficients are plotted. The differences observed are rather important. If we increase ε, then the flow is smooth. The spectrum remains smooth and decays in amplitude until the round-off error is reached. Except for a small time, the spectrum obtained for the smallest value of ε is jagged and contains many modes which have no physical significance. This suggests that the regularization parameter induces drastic modification of the original flow.

4. VORTEX RING PROBLEM

We now consider the problem of the flow induced by a vorticity distribution initially concentrated in a ring. This is a three-dimensional flow for which the vortex methods presented previously have to be extended to account for the stretching term of the Helmholtz equation (1). Two types of method have been developed for this purpose: the vortex filament method[8] and the vortex point method.[9] The main difference between the two methods is that in the first Kelvin's theorem is

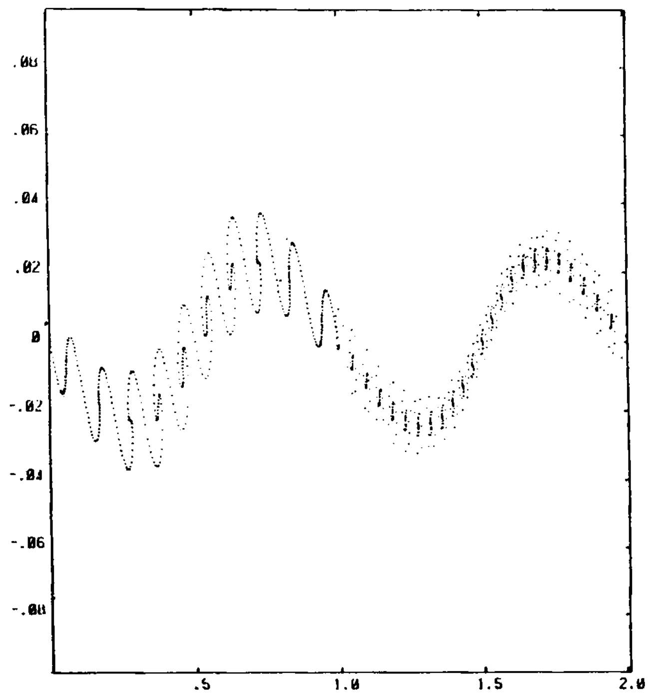
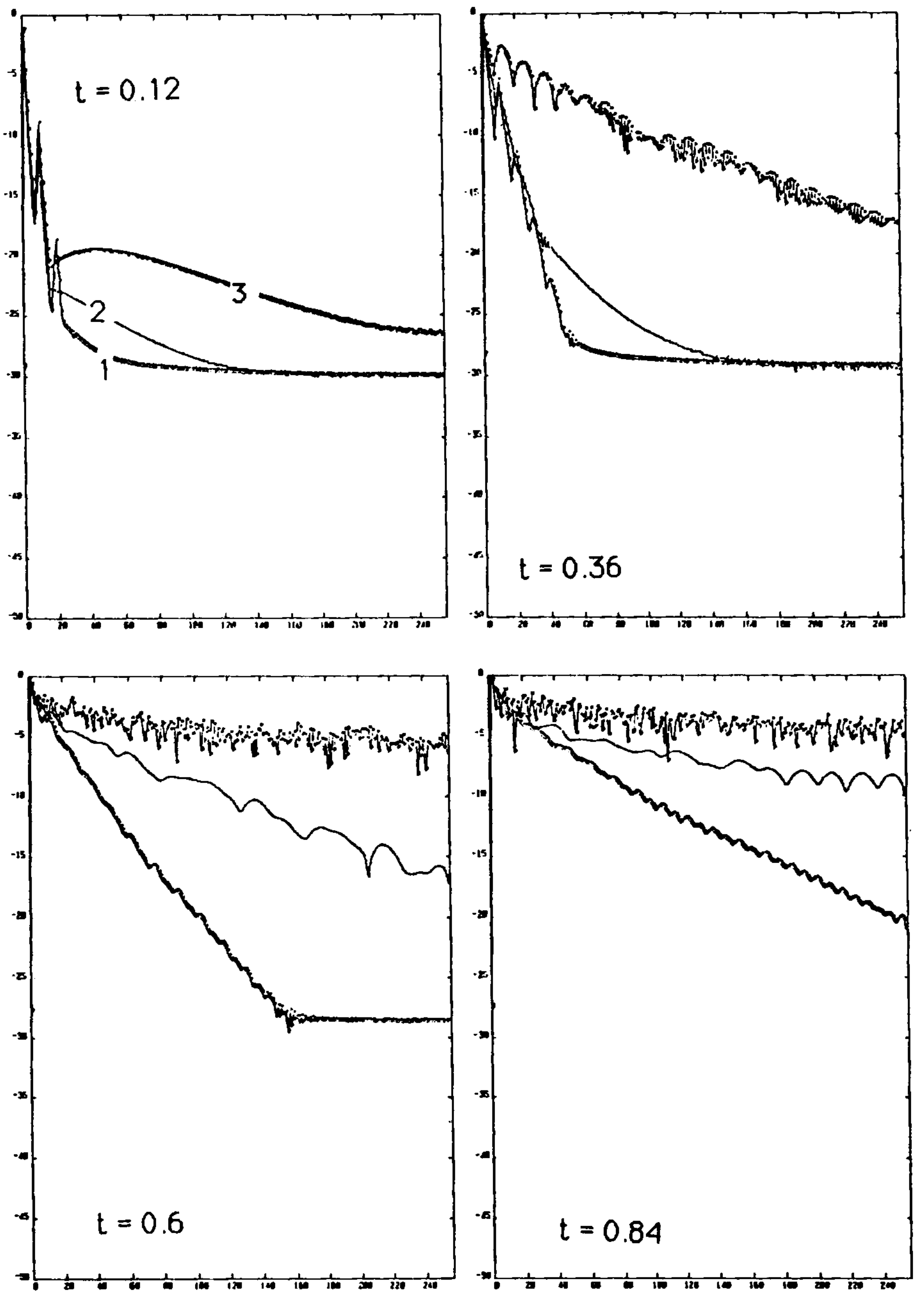


Figure 5(a)



(b)

Figure 5. (a) Point positions at time 0·4 for different values of ε ($\sigma = 0{\cdot}01$, $\sigma_1 = 0{\cdot}0001$, $l_1 = 0{\cdot}1$). (b) Log–linear plots of the amplitudes of the Fourier coefficients ($\sigma = 0{\cdot}01$, $\sigma_1 = 0{\cdot}0001$, $l_1 = 0{\cdot}1$) for different values of ε: 0·1 (curve 1), 0·05 (curve 2), 0·025 (curve 3)

automatically satisfied whereas in the second this theorem is satisfied in an approximate way by integrating the Helmholtz equation. However, we find an advantage in using the vortex point method for this kind of description which remains valid even if the vorticity support has a very complicated shape. Thus in the present work we use the Rehbach method.[9] We recall that the discretized equations are

$$\frac{\mathrm{d}x_k}{\mathrm{d}t}=\sum_i K_\varepsilon(x_k(t)-x_i(t))\alpha_i(t), \tag{5}$$

$$\frac{\mathrm{d}\alpha_k}{\mathrm{d}t}=\alpha_k\nabla\left(\sum_i K_\varepsilon(x_k(t)-x_i(t))\alpha_i(t)\right). \tag{6}$$

The technique used to discretize the stretching equation (4) is simply a direct derivation of the Biot–Savart law in order to compute the velocity gradient. Then the regularization must be one order better than in the two-dimensional case since this relation includes a one-order-higher singularity.

Another technique can be used to obtain an approximate equation for the stretching. In the continuous problem we have $\omega\cdot(\nabla u)=\omega\cdot(\nabla^{\mathrm{T}}u)$, where superscript T denotes transposition. Then, in the discretization of equation (4) we can use any combination of the velocity gradient and the transposed velocity gradient:

$$\omega\cdot(\nabla u)=\omega\cdot[(1-\alpha)\nabla^{\mathrm{T}}u+\alpha\nabla u].$$

Thus we have the new scheme

$$\frac{\mathrm{d}\alpha_k}{\mathrm{d}t}=\alpha_k\sum_i[\alpha\nabla(K_\varepsilon(x_k(t)-x_i(t))\alpha_i(t))+(1-\alpha)\nabla^{\mathrm{T}}(K_\varepsilon(x_k(t)-x_i(t))\alpha_i(t))].$$

An important feature of this new algorithm is the use of the transposed velocity gradient. It has been pointed out by Choquin and Cottet[10] that the direct formulation ($\alpha=1$) does not conserve the total vorticity whereas the discrete equation in which the stretching is expressed only in terms of the transposed velocity gradient ($\alpha=0$) is conservative.

In the original work of Rehbach, two values of α, namely $\alpha=1$ and $\alpha=0{\cdot}5$, were used, and the latter was found to have better numerical properties. It will be a goal of our numerical test to determine the effect of this value on the stability of the computed solution.

In our calculation we choose the initial vorticity to fit the steady solution of Norbury.[11] We use a set of 320 particles comprised of 80 sections containing four particles each.

We first try to compute the Norbury steady solution in order to study the effect of the regularization parameter ε and the parameter α. We observe that for $\alpha=1$ (direct formulation) no solution can be obtained for a long time (Figure 6(a)). It is possible to adapt the computation by increasing the smoothing parameter ε (Figure 6(b)) as in the previous section it is found that a large value of this parameter enables the computation to be carried for a longer time. However, the perturbations due to round-off error, whose early development can be assumed to be physically consistent, stimulate some numerical instabilities associated with the stretching term discretization. This appears clearly if we compute the global angular momentum, which should be zero. We can observe in Figure 6(c) that it grows rapidly even when ε is increased. This can be explained by two different reasons.

The first is related to the results of the linear stability analysis of the vortex ring.[12] It is found that the more unstable modes are those of short wave number. This is the case in our calculations where the breakdown of the vortex ring always occurs between the first and last initialized sections. We can conjecture that it is a result of the accumulation of round-off errors.

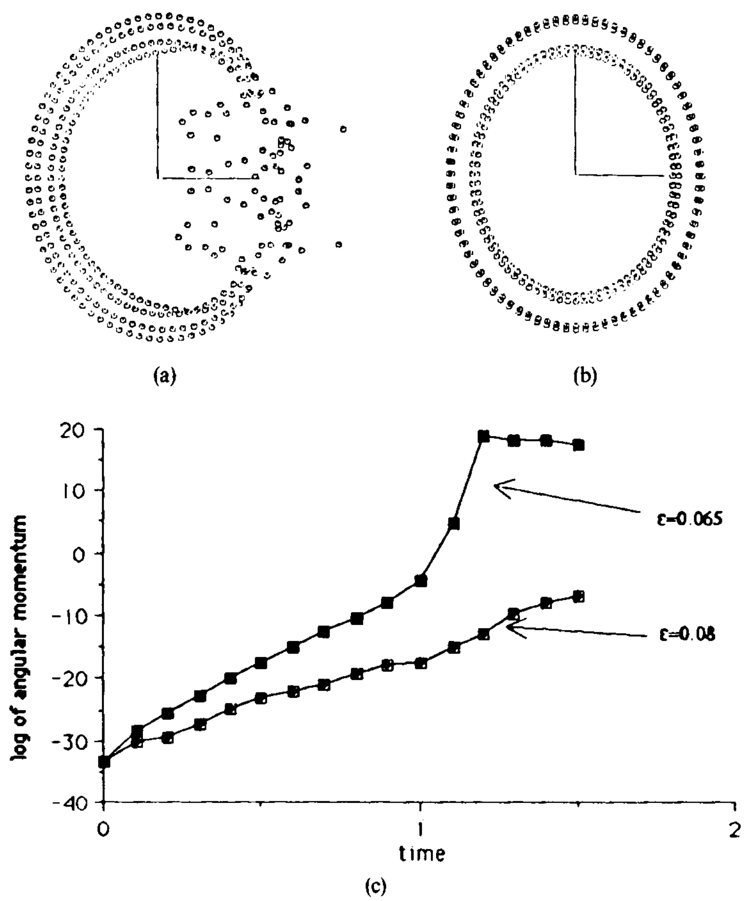


Figure 6. (a) Point positions at time 1·1 for $\varepsilon = 0{\cdot}065$, $\alpha = 1$. (b) Points positions at time 1·1 for $\varepsilon = 0{\cdot}082$, $\alpha = 1$. (c) Growth of the angular momentum for $\alpha = 1$

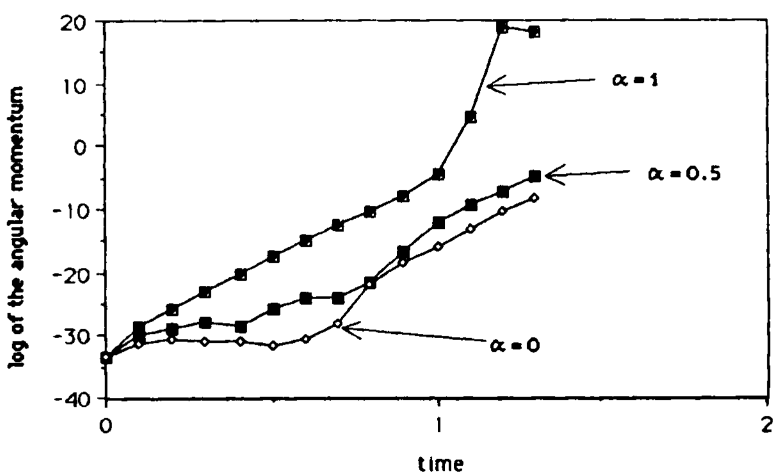
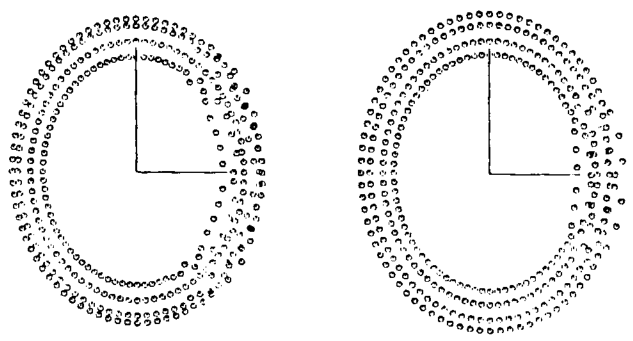


Figure 7. Growth of the angular momentum for different values of α (ε=0·065)



(a) (b)

Figure 8. (a) Point positions at time 1·1 for $\varepsilon = 0{\cdot}065$, $\alpha = 0{\cdot}5$. (b) Point positions at time 1·1 for $\varepsilon = 0{\cdot}065$, $\alpha = 0$

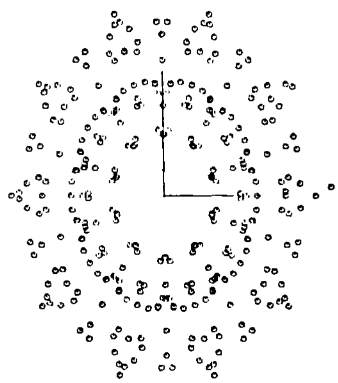


Figure 9. Point positions at time 1·1 with a radial perturbation for $\varepsilon = 0{\cdot}065$, $\alpha = 1$

The second is related to the particle discretization. In the stability analysis, Kelvin's theorem implies that the motion of a ring element and the stretching of the vorticity corresponding to this element are coupled. In the particle method these two aspects of the ring evolution are computed independently and thus have different evolutions. The result is that Kelvin's theorem is not automatically satisfied and this leads to numerical instability, at least for the case $\alpha = 1$. This seems to be confirmed by the fact that no such problem has been encountered by Ghoniem *et al.*,[13] who compute similar flows using vortex filament discretization.

If now we use the formulation proposed by Rehbach ($\alpha = 0{\cdot}5$) or Choquin and Cottet ($\alpha = 0$), we see in Figure 7 that the growth of the angular momentum is not as marked. Moreover, for the last case ($\alpha = 0$) the vorticity is conserved. Figure 8 shows that the vortex ring does not break down, although the same kind of instability associated with the initial discretization can be observed.

Another way to damp the numerical instability is to superimpose on the initial condition a small radial perturbation. This perturbation affects the evolution of the smaller-wavelength perturbations and allows the computation to be carried out for a longer time. A result of this kind is presented on Figure 9.

5. CONCLUSIONS

In this work we have tried to analyse some computational experiments on interactions between numerical and physical instabilities. In two dimensions it is pointed out that the choice of a smoothing technique will allow computation of the flow for a long time. In three dimensions the additional difficulty associated with vortex stretching has partially been overcome by using a new approximation of this term. The question remains in 3D of knowing whether the computed solution corresponds to a physical rather than a numerical evolution of the perturbations induced by round-off errors.

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